

On the Edge-Balanced Index Sets of Complete Even Bipartite Graphs

Ha Dao^{*} Hung Hua[†] Michael Ngo[‡] Christopher Raridan[§]

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Abstract

In 2009, Kong, Wang, and Lee introduced the problem of finding the edge-balanced index sets (*EBI*) of complete bipartite graphs $K_{m,n}$, where they examined the cases $n = 1, 2, 3, 4, 5$ and the case $m = n$. Since then the problem of finding $EBI(K_{m,n})$, where $m \geq n$, has been completely resolved for the $m, n = \text{odd, odd and odd, even}$ cases. In this paper we find the edge-balanced index sets for complete bipartite graphs where both parts have even cardinality.

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1 Introduction

Given a graph G , let V and E denote the vertex set and edge set of G , respectively. A *binary edge-labeling* of G is a function $f : E \rightarrow \{0, 1\}$. For $i \in \{0, 1\}$, we call $e \in E$ an *i-edge* if $f(e) = i$. Let $e(i)$ denote the number of *i*-edges under a binary edge-labeling f . If $|e(1) - e(0)| \leq 1$, we say that f is *edge-friendly*. Under f , we let $\deg_i(v)$ denote the number of *i*-edges incident with $v \in V$. If f is edge-friendly, then f will induce

^{*}Undergraduate Student, Clayton State University, (hdao@student.clayton.edu)

[†]Undergraduate Student, Georgia Institute of Technology, (hhua7@gatech.edu)

[‡]Undergraduate Student, Clayton State University, (mngo4@student.clayton.edu)

[§]Department of Mathematics, Clayton State University, (ChristopherRaridan@clayton.edu)

a (possibly partial) *vertex-labeling* where v is labeled 1 if $\deg_1(v) > \deg_0(v)$, labeled 0 if $\deg_0(v) > \deg_1(v)$, and unlabeled if $\deg_1(v) = \deg_0(v)$. Any vertex labeled i is called an i -*vertex* and let $v(i)$ denote the number of i -vertices under an edge-friendly labeling f . The *edge-balanced index set* of G is defined as

$$EBI(G) = \{|v(1) - v(0)| : f \text{ is edge-friendly}\}$$

and an element in $EBI(G)$ will be called a *balanced index*. More information about graph labelings, including many results concerning edge-friendly labelings, can be found in Gallian's dynamic survey [1].

Kong, Wang, and Lee [4] explored the problem of finding the edge-balanced index sets of complete bipartite graphs $K_{m,n}$ by investigating the cases where $n = 1, 2, 3, 4$, and 5, as well as the case where $m = n$. In [5], Krop, Minion, Patel, and Raridan concluded the EBI problem for complete bipartite graphs with both parts of odd cardinality (the “odd/odd” case). The following year, Hua and Raridan [2] found $EBI(K_{m,n})$ where $m > n$, m is odd and n is even (the “odd/even” case). In this paper we find the edge-balanced index sets for complete even bipartite graphs (the “even/even” case).

2 Notations and Conventions

Throughout the rest of this paper, we let $K_{m,n}$ denote a complete bipartite graph with part $A = \{v_1, v_2, \dots, v_m\}$ of even cardinality m and part $B = \{u_1, u_2, \dots, u_n\}$ of even cardinality n , where $m \geq n \geq 2$. For any edge-friendly labeling of $K_{m,n}$, we have that $e(0) = e(1) = \frac{mn}{2}$. Without loss of generality, we may assume that labelings are chosen so that $v(1) \geq v(0)$. Let $v_A(i)$ and $v_B(i)$ represent the number of i -vertices in parts A and B , respectively, and note that $v(i) = v_A(i) + v_B(i)$.

For integers $a < b$ define $[a, b] = \{a, a+1, \dots, b\}$ and $[a, a] = \{a\}$. If a is positive, $[a] = \{1, 2, \dots, a\}$; otherwise $[a] = \emptyset$. We organize the edge labels of an edge-friendly labeling of $K_{m,n}$ as an $n \times m$ binary matrix whose (s, t) -entry is the label on edge $u_s v_t$, where $s \in [n]$ and $t \in [m]$. The vertex label for $v_t \in A$ or $u_s \in B$ is found by comparing the sum of the entries in column t or row s with $\frac{n}{2}$ or $\frac{m}{2}$, respectively.

Example 2.1 *Finding the corresponding balanced index for each of the following edge-friendly labelings of $K_{4,4}$ is straightforward: (a) and (b) show two different edge-*

friendly labelings that each give $0 \in EBI(K_{4,4})$, (c) shows that $1 \in EBI(K_{4,4})$, and (d) shows that $2 \in EBI(K_{4,4})$.

1	1	1	1	1	1	1	1	1	0	1	1	1	0
1	1	1	1	1	1	1	1	0	1	1	1	1	0
0	0	0	0	1	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
(a)				(b)				(c)				(d)	

Figure 1: Some edge-friendly labelings of $K_{4,4}$.

The quotient and remainder when x is divided by y using the division algorithm is denoted by $x \operatorname{div} y$ (or, $\lfloor \frac{x}{y} \rfloor$) and $x \operatorname{mod} y$, respectively.

3 Finding $EBI(K_{m,n})$

In this section, we prove

Theorem 3.1 *Let $K_{m,n}$ be a complete bipartite graph with parts of cardinality m and n , where $m \geq n$ are positive even integers. Then $EBI(K_{m,2}) = \{0\}$. For $n \geq 4$, let $k = \lfloor \frac{mn}{n+2} \rfloor$, $k' = \frac{mn}{2} \operatorname{mod} (\frac{n}{2} + 1)$, $j = \lfloor \frac{mn}{m+2} \rfloor$, and $j' = \frac{mn}{2} \operatorname{mod} (\frac{m}{2} + 1)$. Then*

$$EBI(K_{m,n}) = \begin{cases} \{0, 1, \dots, 2(k+j) + 2 - m - n\}, & \text{if } k' = \frac{n}{2} \text{ and } j' = \frac{m}{2}, \\ \{0, 1, \dots, 2(k+j) + 1 - m - n\}, & \text{if either } k' = \frac{n}{2} \text{ or } j' = \frac{m}{2}, \\ \{0, 1, \dots, 2(k+j) - m - n\}, & \text{if } k' < \frac{n}{2} \text{ and } j' < \frac{m}{2}. \end{cases}$$

Proof. In [4], the authors show that $EBI(K_{m,2}) = \{0\}$ for all integers $m \geq 2$. Throughout the rest of this paper, we assume that $m \geq n \geq 4$ are both even.

To find the maximal element of $EBI(K_{m,n})$ we need an edge-friendly labeling that maximizes the value of $v(1)$ while at the same time minimizes the value of $v(0)$. Let k and j represent the maximum value of $v_A(1)$ and $v_B(1)$, respectively. A 1-vertex $v \in A$ must have $\deg_1(v) \geq \frac{n}{2} + 1$, so the maximum value of $v_A(1)$ is $k = e(1) \operatorname{div} (\frac{n}{2} + 1) = \lfloor \frac{mn}{n+2} \rfloor$. Given any edge-friendly labeling that maximizes $v_A(1)$ where each of the 1-vertices in A has exactly $(\frac{n}{2} + 1)$ 1-edges, the number of 1-edges incident with the other $(m - k)$ vertices in A is $k' = e(1) \operatorname{mod} (\frac{n}{2} + 1)$. If $v_A(1)$ is maximized and $k' < \frac{n}{2}$, there are not enough of these extra 1-edges to

create an unlabeled vertex in A , so $v_A(0) = m - k$. If instead we have that $k' = \frac{n}{2}$, there are enough extra 1-edges to allow an unlabeled vertex in part A and having an unlabeled vertex in A reduces the value of $v_A(0)$, which would be $m - k - 1$ in this case. Similarly, the maximum value of $v_B(1)$ is $j = e(1) \operatorname{div} \left(\frac{m}{2} + 1 \right) = \lfloor \frac{mn}{m+2} \rfloor$. For any edge-friendly labeling that maximizes $v_B(1)$ where each of the 1-vertices in B has exactly $\left(\frac{m}{2} + 1 \right)$ 1-edges, the number of 1-edges incident with the other $(n - j)$ vertices in B is $j' = e(1) \bmod \left(\frac{m}{2} + 1 \right)$. When $v_B(1)$ is maximized, if $j' < \frac{m}{2}$, then $v_B(0) = n - j$, and if $j' = \frac{m}{2}$, then $v_B(0) = n - j - 1$.

Now, we need to find an edge-friendly labeling that maximizes both $v_A(1)$ and $v_B(1)$ at the same time, thus maximizing their sum $v(1)$. Maximizing $v(1)$ and allowing part A or part B to contain an unlabeled vertex (when $k' = \frac{n}{2}$ or $j' = \frac{m}{2}$) minimizes both $v_A(0)$ and $v_B(0)$ at the same time, thus minimizing their sum $v(0)$. That is,

$$\max EBI(K_{m,n}) = \begin{cases} 2(k + j) + 2 - m - n, & \text{if } k' = \frac{n}{2} \text{ and } j' = \frac{m}{2}, \\ 2(k + j) + 1 - m - n, & \text{if either } k' = \frac{n}{2} \text{ or } j' = \frac{m}{2}, \\ 2(k + j) - m - n, & \text{if } k' < \frac{n}{2} \text{ and } j' < \frac{m}{2}. \end{cases}$$

For example, if $m = n = 4$, then $k = j = k' = j' = \frac{m}{2} = \frac{n}{2} = 2$ and $\max EBI(K_{4,4}) = 2$. Example 2.1(d) shows an edge-friendly labeling for $K_{4,4}$ that produces the maximal balanced index for this graph. For all other values of even integers $m \geq n$, it follows that $k > \frac{m}{2}$ and $j > \frac{n}{2}$, which ensures that in each of the three cases above, $\max EBI(K_{m,n})$ is a positive quantity.

We now discuss an algorithm that provides a sequence of edge-friendly labelings (actually, a sequence of edge-label switches) that correspond to each of the balanced indices from 0 to $\max EBI(K_{m,n})$. For each of the following steps, we mention only the vertices whose labels have changed due to the switches described in that step. For some values of m and n , running the entire algorithm is unnecessary; indeed, the procedure should be terminated when $\max EBI(K_{m,n})$ has been obtained. We will provide a few example graphs when early termination is allowed.

Step 0. For $s \in \left[\frac{n}{2} \right]$ and $t \in [m]$, create an $n \times m$ matrix whose (s, t) -entry is 1 and set all other entries to 0. The top half of this matrix is all 1s and the bottom half is all 0s, so $v_A(1) = v_A(0) = 0$, $v_B(1) = v_B(0) = \frac{n}{2}$, and $0 \in EBI(K_{m,n})$.

In Steps 1-3, we let $q(a)$ and $r(a)$ represent the quotient and remainder, respectively, when $a - 1$ is divided by $\frac{n}{2}$ using the division algorithm.

Step 1. For $a \in [\frac{m}{2} - 1]$, switch the $(\frac{n}{2} + 1, a)$ -entry with the $(\frac{n}{2} - r(a), m - q(a))$ -entry. Here, each edge-label switch exchanges a 0 in row $\frac{n}{2} + 1$ with a 1 in the last $q(\frac{m}{2} - 1) + 1$ columns and above the $(\frac{n}{2} + 1)$ -st row. Such switches cause v_a to become a 1-vertex for $a \in [\frac{m}{2} - 1]$ and v_{m+1-b} to become a 0-vertex for $b \in [q(\frac{m}{2} - 1) + 1]$. Note that the first switch of a 1 in each column has no effect on the balanced index since both $v_A(1)$ and $v_A(0)$ increase by 1, but that each subsequent switch will increase the balanced index by 1. At the end of Step 1, $\deg_0(u_{\frac{n}{2}+1}) = \frac{m}{2} + 1$, which means $u_{\frac{n}{2}+1}$ is just “barely” a 0-vertex.

Step 2. Switch the $(\frac{n}{2} + 1, \frac{m}{2})$ -entry with the $(\frac{n}{2} - r(\frac{m}{2}), \frac{m}{2})$ -entry. This switch causes vertex $u_{\frac{n}{2}+1}$ to become an unlabeled vertex so the balanced index increases by 1. Now, switch the $(\frac{n}{2} - r(\frac{m}{2}), \frac{m}{2})$ -entry with the $(\frac{n}{2} - r(\frac{m}{2}), m - q(\frac{m}{2}))$ -entry, which causes $v_{\frac{m}{2}}$ to become a 1-vertex. If $r(\frac{m}{2}) = 0$, or equivalently $m = tn + 2$ for some integer $t \geq 1$, then this switch also causes $v_{m-q(\frac{m}{2})}$ to become a 0-vertex and there is no change in the balanced index; otherwise, $v_{m-q(\frac{m}{2})}$ was already a 0-vertex and the balanced index increases by 1. Note that for $m = n = 4$, we terminate the procedure since $k = \frac{m}{2}$ and $j = \frac{n}{2}$ for this case.

For other values of m and n , the $(\frac{m}{2} + 1)$ -st (double) switch is similar to the $\frac{m}{2}$ -th. Exchange the $(\frac{n}{2} + 1, \frac{m}{2} + 1)$ -entry with the $(\frac{n}{2} - r(\frac{m}{2} + 1), \frac{m}{2} + 1)$ -entry. This switch causes $u_{\frac{n}{2}+1}$ to become a 1-vertex so the balanced index increases by 1. Now, switch the $(\frac{n}{2} - r(\frac{m}{2} + 1), \frac{m}{2} + 1)$ -entry with the $(\frac{n}{2} - r(\frac{m}{2} + 1), m - q(\frac{m}{2} + 1))$ -entry, which causes $v_{\frac{m}{2}+1}$ to become a 1-vertex. If $r(\frac{m}{2} + 1) = 0$, or equivalently $m = tn$ for some integer $t \geq 1$, then this switch also causes $v_{m-q(\frac{m}{2}+1)}$ to become a 0-vertex and there is no change in the balanced index. Otherwise, $v_{m-q(\frac{m}{2}+1)}$ was already a 0-vertex and the balanced index increases by 1. Note that if $(m, n) = (6, 4)$, $(8, 4)$, $(10, 4)$, or $(6, 6)$, then we terminate the procedure since $v_A(1) = k = \frac{m}{2} + 1$, $v_B(1) = j = \frac{n}{2} + 1$, and $j' < \frac{m}{2}$ since maximizing $v_B(1)$ does not allow for an unlabeled vertex in part B for these cases.

Step 3. Perform this step if and only if $k > \frac{m}{2} + 1$. For $a \in [\frac{m}{2} + 2, k]$, switch the $(\frac{n}{2} + 1, a)$ -entry with the $(\frac{n}{2} - r(a), m - q(a))$ -entry. This step is essentially the same as Step 1, just applied to a different set of indices. Moreover, we have now forced $v_A(1) = k$ and either $v_A(0) = m - k$ (there are no unlabeled vertices in part A) or $v_A(0) = m - k - 1$ (there is one unlabeled vertex in A). That is, we have maximized

$v_A(1)$ and minimized $v_A(0)$. Additionally, $v_B(1) = \frac{n}{2} + 1$ and $v_B(0) = n - v_B(1)$, so if $j = \frac{n}{2} + 1$ and $j' < \frac{m}{2}$, then terminate the procedure.

Step 4. Perform this step if and only if $j > \frac{n}{2} + 1$ or $j' = \frac{m}{2}$. In this step, we only make switches with entries that are in the same column, thereby preserving the current vertex labels for all of the vertices in part A . Since Steps 1-3 exchange k 1s in the top half of the matrix for 0s from row $\frac{n}{2} + 1$, we have $\deg_1(u_s) = k + 1 > \frac{m}{2} + 2$ for $s \in [c]$, where $c = \frac{n}{2} - k \bmod \frac{n}{2}$ is a positive integer. For these c 1-vertices in part B , we may replace up to $d = (k + 1) - (\frac{m}{2} + 1) = k - \frac{m}{2} > 0$ of their incident 1-edges with 0-edges and the vertices will remain 1-vertices. Now, $\deg_1(u_s) = k > \frac{m}{2} + 1$ for $s \in [\frac{n}{2} + 1 - k \bmod \frac{n}{2}, \frac{n}{2} + 1]$, so for these $(1 + k \bmod \frac{n}{2})$ 1-vertices in B , we may replace up to $d - 1 > 0$ of their incident 1-edges with 0-edges and the vertices will remain 1-vertices. Moreover, $\deg_1(u_s) = 0$ for $s \in [\frac{n}{2} + 2, n]$.

Let b be the total number of switches that we need to perform to obtain the maximal balanced index. If $j' = \frac{m}{2}$, then part B should contain an unlabeled vertex and $b = \frac{mn}{2} - (\frac{m}{2} + 1)(\frac{n}{2} + 1)$; otherwise, $b = \left\lceil j - (\frac{n}{2} + 1) \right\rceil (\frac{m}{2} + 1)$. For $a \in [cd]$, switch the

$$\left(1 + (a - 1) \operatorname{div} d, 1 + (a - 1) \bmod \left(\frac{m}{2} + 1\right)\right)\text{-entry}$$

with the

$$\left(\frac{n}{2} + 2 + (a - 1) \operatorname{div} \left(\frac{m}{2} + 1\right), 1 + (a - 1) \bmod \left(\frac{m}{2} + 1\right)\right)\text{-entry}.$$

This collection of switches takes the extra d 1s on row s , where $s \in [c]$, and exchanges them for 0s that are in the same column but on a different row (and in the bottom half of the matrix). The row that is losing 0s and gaining 1s will continue to do so until the number of 1s in that row is $\frac{m}{2} + 1$, at which time the procedure simply moves down to the next row and “starts over” (due to the mod operator). When the number of 1s in a row reaches $\frac{m}{2}$, the corresponding vertex changes from a 0-vertex to an unlabeled vertex and the balanced index increases by 1. Similarly, when the number of 1s in a row reaches $\frac{m}{2} + 1$, the unlabeled vertex becomes a 1-vertex and the balanced index increases by 1 again.

Continuing, for $a \in [cd + 1, b]$, switch the

$$\left(1 + c + (a - 1 - cd) \operatorname{div} (d - 1), 1 + (a - 1) \bmod \left(\frac{m}{2} + 1\right)\right)\text{-entry}$$

with the

$$\left(\frac{n}{2} + 2 + (a - 1) \operatorname{div} \left(\frac{m}{2} + 1\right), 1 + (a - 1) \bmod \left(\frac{m}{2} + 1\right)\right)\text{-entry.}$$

This collection of switches is similar to those just completed, except we are taking only $d - 1$ extra 1s on a row and exchanging them with 0s in the same column but on a different row. The balanced index changes as before, as well.

We started with balanced index 0 and every switch described by the algorithm increased the balanced index by at most 1. Although not all graphs required that every step of the algorithm completed, any early termination of the procedure was due to having already obtained $\max EBI(K_{m,n})$. At the end of Step 3, we remarked that $v_A(1) = k$ and either $v_A(0) = m - k - 1$ or $m - k$, according to whether part A did or did not contain an unlabeled vertex. Step 4 does not alter the labels of vertices in part A . Upon completion of Step 4, we find that $v_B(1) = j$ and either $v_B(0) = n - j$ (there are no unlabeled vertices in part B) or $v_B(0) = n - j - 1$ (there is one unlabeled vertex in B). Thus, the construction provided by the algorithm produces the maximal balanced index for $K_{m,n}$, where $m \geq n \geq 2$ are even integers. ■

The first author has written a MATLAB m-file that shows the output of each edge-label switch described in Steps 1-4. The file, called `EBI_K.m`, can be found at <http://www.clayton.edu/faculty/craridan/code>.

References

- [1] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 5:DS 6, <http://www.combinatorics.org/ojs/index.php/eljc/index>, 2015.
- [2] Hung Hua and Christopher Raridan. On the edge-balanced index sets of odd/even complete bipartite graphs. *Cong. Numer.*, volume 219, pages 227–232, 2014.
- [3] Man C. Kong and Sin-Min Lee. On edge-balanced graphs. In *Graph Theory, Combinatorics, and Algorithms, Vol. 1,2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., pages 711–722. Wiley, New York, 1995.
- [4] Man C. Kong, Yung-Chin Wang, and Sin-Min Lee. On edge-balanced index sets of some complete k -partite graphs. In *Proceedings of the Fortieth Southeastern*

International Conference on Combinatorics, Graph Theory and Computing, volume 196, pages 71–94, 2009.

- [5] Elliot Krop, Sarah Minion, Pritul Patel, and Christopher Raridan. A solution to the edge-balanced index set problem for complete odd bipartite graphs. *Bull. Inst. Combin. Appl.*, volume 70 (2014), 119–125.